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Ruin theory in insurance and a comparison of methods for approximating of the ruin probability in an infinite time horizon

Abstract

The article presents the theoretical background of ruin theory and the description of the classical model for the insurer's surplus. Analytical calculations of the ruin probability are presented for special cases of single loss distribution (exponential, gamma and a mixture of exponential distributions). The main focus is on the analysis of available methods for approximation of the ruin probability in an infinite horizon in continuous time model. The quality of approximation is tested by comparing the approximated ruin probability with the probability determined analytically (if possible) or estimated numerically using the Pollaczek-Khinchin formula. The approximation errors (in both absolute and relative terms) are shown for selected light-tailed distributions (mixture of exponential distributions, gamma) and heavy-tailed distributions (Pareto, lognormal, Weibull and Burr). The goal of the article is the assessment of the possibility to use the approximation methods for ruin probability by insurance companies, including areas such as pricing or solvency, especially in the context of Solvency II regime. The conducted analyses show that in most cases approximation results are quite satisfactory (relative error not exceeding 5%) and the lowest errors are observed for Cramer-Lundberg and De Vylder approximations in case of light-tailed distributions and for Beekman-Bowers and De Vylder approximations in case of heavy-tailed distributions. The approximation quality, measured with relative error, in general deteriorates in line with the decreasing assumed ruin probability, especially for heavy-tailed distributions.

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Introduction

The activities of financial institutions, such as insurance companies and banks, inevitably involve risks, which can take various forms. In the case of a bank, it is mainly financial and credit risk; in the case of an insurance company, insurance risk, related to the course of claims among insured persons, plays an important role. Key to the management of an insurance company is the quantification of risk and maintaining it at an appropriate level, consistent with the so-called risk appetite. The realization of insurance risk can carry serious consequences, even leading to the bankruptcy of the company. For this reason, supervisory authorities in many countries issue a number of regulations aimed at minimizing the risk of bankruptcy, including the need to calculate so-called solvency requirements. In the European Union, solvency issues are regulated by the EU Solvency II Directive, which came into force on January 1, 2016. According to this directive, entities must calculate the Solvency Capital Requirement (SCR) and the Minimum Capital Requirement (MCR). The SCR corresponds to the value at risk of an insurance company's basic own funds at a confidence level of 99.5% over a one-year period, and is calculated taking into account six risk modules (market, counterparty default, health underwriting, life underwriting, non-life underwriting, intangible assets), operational risk, and an adjustment for the loss-absorbing capacity of technical provisions and deferred income taxes.

In addition to calculating the solvency requirement to meet regulatory requirements, insurance companies also use other measures that can help manage risk. One such measure is the ruin probability, which can be calculated for a specific portfolio of insurance contracts and tells how much the probability is that over a certain time horizon (finite or infinite) the total value of claims will exceed the total value of premiums plus the so-called initial surplus, i.e. the insurance company will be ruined. Ruin theory provides useful mathematical tools for quantifying the risks faced by a company. In many cases, however, the exact calculation of the ruin probability is difficult or even impossible, and approximations are necessary. Assessing the possibilities of using various approximation methods is the main objective of this article. It is divided into two parts: the first part presents an introduction to the ruin theory, its selected properties, as well as limitations and difficulties. Finally, the so-called classical model is derived, along with its application to several probability distributions of the loss amount with explicit analytical formulas. The second part reviews selected approximation methods for the model with continuous solvency control over an infinite time horizon and their empirical results using selected examples. It concludes with an assessment of the feasibility of using the ruin theory and methods of approximation of ruin probability by insurance companies in areas such as pricing, risk management and in the context of Solvency II regime requirements.

1. Ruin Theory – general theory

1.1. Basic concepts. Continuous insurer's surplus model

In the literature, ruin theory is an essential tool for systematically monitoring the long-term performance of an insurance company. Its analysis should begin by defining the concepts that form the basis for further consideration. Conceptually, the theory focuses on the fact that insurance companies in the basic scope of their activities experience cash flows in two directions. Inflows are income from collected premiums, while the source of the outflows is the amount of claims paid. This ignores the activity of insurance companies in all other areas (e.g. investment, deposits), as well as other sources of costs (administrative, acquisition, other operational, etc.). The difference between collected premiums and paid claims in a given time period is called the surplus – we understand a negative surplus as a deficit. We further assume that the insurance company has some initial surplus at the start, enabling it to begin providing services. In the model defined in this way, we focus exclusively on cashflows related to technical activities, focusing our attention on the occurrence of the risk diversification[1] effect in the time dimension. The mathematical representation of the above description is the long-term surplus process (amount of own funds) of an insurance company, which is the following function of time:

$$U(t) = u + ct - S(t) \tag{1}$$

where:
$t \geq 0$ – the variable expressing consecutive time units,
$u \geq 0$ – initial surplus of the insurance company at time $t = 0$,
$c \geq 0$ – premiums collected in a time unit,
$U(t)$ – the value of the process at time t,
$S(t) = \sum_{i=1}^{N(t)} Y_i$ – the total value of compensation payments for claims incurred in the period $(0, t]$,
Y_i – the value of compensation payments from a single claim,
$N(t)$ – number of claims incurred in the period $(0, t]$.

In the model, we assume that all claims Y_i come from the same probability distribution and are mutually independent. The process of occurrence of claims $N(t)$ is also independent of their value. It does however depend on the number of insured risks. It should be noted that the model is characterized by dual randomness, i.e., we are dealing with a *random number of claims:* $N(t)$ – with *random amounts of compensation*: Y_i.

[1] Diversification effect is one of the key concepts in the insurance industry. According to the statutory definition, it is: "*the reduction in the risk exposure of insurance and reinsurance undertakings and groups related to the diversification of their business, resulting from the fact that the adverse outcome from one risk can be offset by a more favourable outcome from another risk, where these risks are not fully correlated*"; In ruin theory, one may understand the diversification effect as the ability to offset losses incurred in one of the reporting periods in future periods.

Insurer ruin (the event of insolvency) is defined as the first moment of time at which the surplus process presented in equation (1) takes negative value

$$T \coloneqq inf\,(t \geq 0\colon U(t) < 0). \tag{2}$$

Then, at finite t we consider the **ruin probability on a finite time horizon** (specifically, over a time segment of length $(0, t)$), defined as following (Otto 2008):

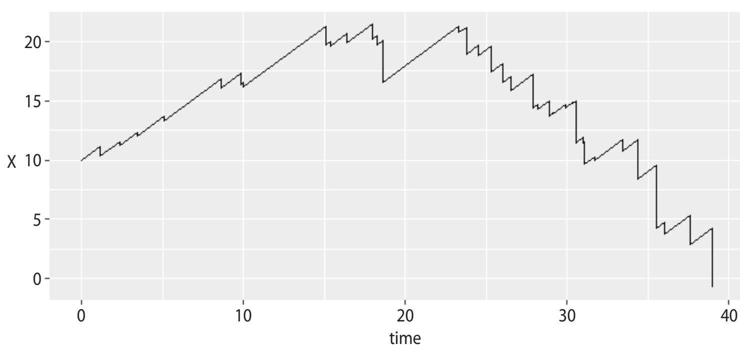
$$\psi(u,t) \coloneqq P(T < t). \tag{3}$$

In contrast, when time tends to infinity, we speak of **ruin probability in an infinite time horizon**, defined below (Otto 2008):

$$\psi(u) \coloneqq \lim_{t\to\infty} \psi(u,t) = P(T < \infty). \tag{4}$$

The presented formulae, encapsule the so-called continuous model of insurer surplus, which assumes that the insurer is subjected to solvency control at every moment of the process, and the ruin of the undertaking is unambiguously determined by the first moment when the value of the process becomes negative. Figure 1 shows an example of the surplus process in a model with continuous solvency control.

Figure 1. Example insurer surplus process U(t) in a model with continuous solvency control



Source: Own study.

The x axis represents time, while the y axis is the initial surplus. In the presented example, the process begins with an initial surplus of u = 10. It is worth noting the constant intensity of the premium inflows, which in the graph is represented by a constant slope in the intervals when the function is increasing (between individual claims). The sudden decreases in the value of the function present the moments of payment of individual claims. The moment $T \approx 38$ (time in this model is a continuous variable with infinite divisibility) is the moment of ruin, as the first drop below zero occurs here. The occurrence of ruin does not necessarily mean that a company becomes bankrupt. Ruin only reflects a negative result on pure technical activity, ignoring all other economic activities of the insurance company (costs and profits).

1.2. Classical model. Adjustment coefficient. Lundberg's inequality

The classical model in ruin theory is constituted by further assumption of an exponential distribution of intervals between consecutive claims. Following this assumption, one can prove that the number of claims in a time unit follows a Poisson distribution. Then the process of occurrence of claims $N(t)$ satisfies the assumptions of a Poisson process (with independent and stationary increments), while the $S(t)$ component appearing in equation (1), representing the total value of claims in a given period, is given by a compound Poisson distribution with parameters $(\lambda t, F_y)$, i.e.: $S(t) = \sum_{i=1}^{N(t)} Y_i \sim CPoisson(\lambda t, F_y)$, where F_y is the cumulative distribution function of the value distribution of a single claim (Niemiro 2013). The assumptions of mutual independence of the amounts of individual claims Y_i and their independence from the variable $N(t)$ remain in force.

Modifying the notation of the classical surplus process, it is possible to make elementary conclusions about the premium in the classical model. The first step is to divide the process into individual segments until the next of the k claims occurs:

$$U(T_k) = u + (cW_1 - Y_1) + (cW_2 - Y_2) + \cdots + (cW_k - Y_k) \tag{5}$$

where:

$W_k = T_k - T_{k-1} \sim Exp(\lambda)$ – the time interval between consecutive claims is described by an exponential distribution,

$E(W_k) = \frac{1}{\lambda}$,
$T_0 = 0$,
$E(Y_k) = \mu$.

Then the process is represented by: unchanged initial surplus (u), while each segment ($cW_k - Y_k$) represents *the surplus between the premium obtained from the time of the (k–1)-th claim, until the k-th claim*. Moreover, we know that all claims follow the same distribution. Thus, the expected value of *the excess premium over a single claim* is:

$$E(cW_k - Y_k) = cE(W_k) - E(Y_k) = cE(W) - E(Y) = \frac{c}{\lambda} - \mu. \tag{6}$$

It can be noted that if the surplus is negative (collected premiums are less than the expected value of claims), ruin occurs with probability equal to 1 – this follows from the strong law of large numbers (Szekli 2012).

$$\text{If } E(cW_k - Y_k) < 0, \text{ then } P\left(\lim_{n\to\infty} \sum_{k=1}^{n} (cW_k - Y_k) = -\infty\right) = 1 \tag{7}$$

This condition thus formally determines the intuitively obvious property that the premium must satisfy – it must exceed the expected value of claims paid.

$$c > \mu\lambda \tag{8}$$

The premium is also often written with an additional factor representing the so-called safety loading. The premium is then determined as the expected claims payment scaled by the mentioned safety loading factor $\theta > 0$.

$$c = (1 + \theta)\mu\lambda, \theta > 0 \tag{9}$$

The positive sign of the θ coefficient is a necessary condition to avoid ruin. In practice, insurance companies want the ruin probability to be low and oscillate within the (tolerable) limits of zero.

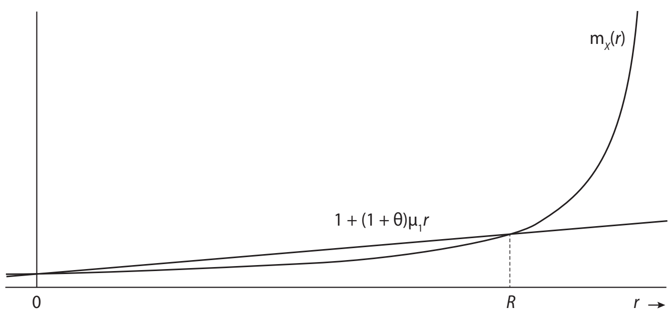
Essential to determine and/or approximate the ruin probability is the so-called **adjustment coefficient**. By introducing the safety loading coefficient, we can define the adjustment coefficient as the positive root of the equation (Kaas et al. 2009):

$$1 + (1 + \theta)\mu \cdot r = M_x(r), \tag{10}$$

where $M_x(r)$ – moment-generating function (later abbreviated MGF).

The coefficient forces the existence of the moment-generating function of the loss distribution, and replicates the assumption of premiums exceeding the expected value of the loss amount (otherwise there is no positive solution to the equation). Under these assumptions, we know that the adjustment coefficient is uniquely determined, which finds its elegant graphical interpretation (Figure 2). The left-hand side of equation (10) is a linear function of variable r with slope $(1 + \theta)\mu$ and intercept 1. The right-hand side – $M_x(r)$ is a convex function on the interval $(0, \infty)$ – this is determined by the sign of the second derivative – $M''_x(r) = E(X^2 e^{rX}) > 0$. Moreover, the first derivative of the MGF at 0 is less than the slope of the left-hand side of the equation, $M'_x(0) < (1 + \theta)\mu$. This implies that the graphs of both sides of the equality must intersect at exactly one point on the positive half-axis, uniquely determining the adjustment coefficient.

Figure 2. Uniqueness of existence of the adjustment coefficient



Source: Kaas et al. 2009, 92.

With the coefficient R at our disposal, we can finally establish the formula for the ruin probability in the classical model (Grandell 1991):

$$\psi(u) = \frac{e^{-Ru}}{E(e^{-RU(T)}|T < \infty)}, u \geq 0. \quad (11)$$

Unfortunately, its practical use tends to be problematic. Particular difficulties are posed by the denominator of the right-hand side of the equation, where it is necessary to calculate the conditional expected value of a process surplus function given the ruin actually occurs. In essence, this is a moment-generating function of the distribution of the amount of deficit at the time of the occurrence of a loss, which is not explicitly known. Calculating this expression in general is difficult, often requiring solving differential equations or using the Laplace transform (Shortle et al. 2003). Taking this into consideration, using reasonable approximations/constraints is common practice. Lundberg's inequality is one of the most popular upper approximations of the ruin probability. It can be justified by a very simple observation: if ruin occurs, then by definition the surplus process reaches a negative value (otherwise there is no ruin). Thus, the denominator of the formula for the ruin probability has to be greater than or equal to 1:

$$E\left(e^{-RU(T)}\middle|T < \infty\right) \geq 1 \quad (12)$$

which leads to the conclusion:

$$\psi(u) \leq e^{-Ru}. \quad (13)$$

Thus, one can essentially omit the denominator of the equation (assuming an extremum of 1 as its value), which must be greater or equal to 1. It is clear that in practice such an estimate can lead to a significant overestimation of the ruin probability if the actual value of the denominator is significantly larger than 1.

1.3. Classical model with loss amount distributions: exponential, n-exponential, gamma

The classical model imposes strict assumptions related to the timing of claims. However, the distribution of the loss amount itself remains free. We know from the formulae for the adjustment coefficient that for it to exist, the moments of the distribution function must be finite, i.e., the MGF has to be specified. While it is difficult to apply the exact formula in the general case, for specific distributions (e.g. two-point, exponential, gamma), exact analytical formulae for the ruin probability can be obtained. The following fragment of this article presents a tabular comparison of how modifying individual input parameters (i.e. loss distribution parameter(s), safety loading coefficient and initial surplus) affects the ruin probability for three light-tailed distributions[2]: exponential, a mixture of exponential distributions and gamma.

[2] The distribution $F_x(x)$ is a light-tailed distribution, if there exist positive a and b such as, for all $x \geq 0$ we have: $1 - F_x(x) \leq ae^{-bx}$. Heavy-tailed distributions are distributions whose tails are not exponentially bounded, i.e. they have thicker tails than the exponential distribution (Rolski 2010).

Assuming that the amount of a single loss follows an exponential distribution, we can take advantage of its so-called "memorylessness" property, which makes the distribution of the deficit at the time of ruin also exponential (Otto 2008). Then, the denominator of formula (11) can be written as a moment-generating function of the exponential distribution, and then simplify the formula for the ruin probability. Further, by calculating the adjustment coefficient for the exponential distribution, we obtain an explicit formula for the ruin probability in the classical model over an infinite time horizon with an exponential loss distribution with parameter β (Otto 2008):

$$\psi(u) = \frac{1}{1+\theta} exp\left(\frac{-\theta\beta u}{1+\theta}\right). \quad (14)$$

For the above set of assumptions, the parameters affecting the ruin probability are u, θ oraz β, i.e. the initial surplus, the safety loading and the exponential distribution parameter respectively. Table 1 shows the ruin probabilities for sample values of the above variables.

Table 1. Ruin probability depending on the initial surplus and safety loading coefficient, with parameter β = 0.001

	***u* = 0**	***u* = 1000 = *μ***	***u* = 5000 = 5*μ***	***u* = 10 000 = 10*μ***	***u* = 25 000 = 25*μ***	***u* = 50 000 = 50*μ***	***u* = 100 000 = 100*μ***
θ = 0.05	0.9524	0.9081	0.7506	0.5916	0.2896	0.0881	0.0081
θ = 0.10	0.9091	0.8301	0.5770	0.3663	0.0937	0.0097	$1.02 \cdot 10^{-4}$
θ = 0.15	0.8696	0.7632	0.4530	0.2360	0.0334	0.0013	$1.88 \cdot 10^{-6}$
θ = 0.20	0.8333	0.7054	0.3622	0.1574	0.0129	$2 \cdot 10^{-4}$	$4.81 \cdot 10^{-8}$
θ = 0.25	0.8000	0.6550	0.2943	0.1083	0.0054	$3.63 \cdot 10^{-5}$	$1.65 \cdot 10^{-9}$
θ = 0.30	0.7692	0.6107	0.2426	0.0765	0.0024	$7.5 \cdot 10^{-6}$	$7.31 \cdot 10^{-11}$

Source: Own study.

For an exponential distribution with β = 0.001, and therefore an expected value of the average loss equal to 1000, in columns one can see how increasing the initial surplus lowers the ruin probability. For example, for a value θ = 0.10, initial surplus in the amount of the expected value of loss results in ruin with a probability of about 83%. Having respectively 5, 10, 25, 50 times the starting capital in relation to the value of the expected amount of loss successively reduces the risk of ruin to less than 1%. Row by row, this effect (for a fixed initial surplus) can be traced in relation to the safety loading coefficient (the insurance company's margin). It is clear that raising the safety loading reduces the risk of ruin by increasing the intensity of premium inflows, i.e. a higher slope of the process on the graph between consecutive claims on increasing segments.

For a mixture of two exponential distributions with parameters α, β and weights q, 1–q respectively, an explicit analytical formula can be obtained using the Laplace transform (Burnecki, Mista, Veron 2005a).

$$\psi(u) = \frac{1}{(1+\theta)(r_2 - r_1)}\{(\rho - r_1)e^{(-r_1 u)} + (r_2 - \rho)e^{(-r_2 u)}\}, \text{ where:}$$

$$r_1 = \frac{\rho + \theta(\alpha+\beta) - [\{\rho + \theta(\alpha+\beta)\}^2 - 4\alpha\beta\theta(1+\theta)]^{\frac{1}{2}}}{2(1+\theta)},$$

$$r_2 = \frac{\rho + \theta(\alpha+\beta) + [\{\rho + \theta(\alpha+\beta)\}^2 - 4\alpha\beta\theta(1+\theta)]^{\frac{1}{2}}}{2(1+\theta)}$$

$$p = \frac{q\alpha^{-1}}{q\alpha^{-1} + (1-q)\beta^{-1}}, \quad \rho = \alpha(1-p) + \beta p. \tag{15}$$

Table 2. Ruin probability depending on the initial surplus and safety loading coefficient for a mixture of exponential distributions with parameters α = 0.001, β = 0.000001 and weights: 0.75; 0.25

	$u = 10^4$	$u = 10^5$	$u = 10^6$	$u = 10^7$	$u = 10^8$
$\theta = 0.05$	0.9518	0.9477	0.9078	0.5907	0.0080
$\theta = 0.10$	0.9080	0.9006	0.8297	0.3653	$9.91 \cdot 10^{-5}$
$\theta = 0.15$	0.8681	0.8579	0.7627	0.2351	$1.82 \cdot 10^{-6}$
$\theta = 0.20$	0.8315	0.8191	0.7048	0.1567	$4.62 \cdot 10^{-8}$
$\theta = 0.25$	0.7979	0.7836	0.6543	0.1077	$1.57 \cdot 10^{-9}$
$\theta = 0.30$	0.7669	0.7511	0.6070	0.0761	$6.93 \cdot 10^{-11}$

Source: Own study.

The parameters of the exponential distributions used in Table 2 were chosen arbitrarily in the above example. A mixture of exponential distributions allows much more flexibility in creating distributions, but requires the estimation of more parameters (in the two-dimensional case, these would be the two parameters of the distributions and their weights). For a mixture of $n > 2$ exponential distributions, it is also possible to derive explicit analytical formulae for the ruin probability (in general, this is possible for phase-type distributions). However, they will not be presented further here due to the progressive complexity of the formulae and the relatively uncommon practical application due to the increasing number of parameters.

Grandell and Segerdahl proved that for losses from a gamma distribution with parameters $\alpha \le 1$ and mean equal to 1, the explicit formula for the ruin probability takes the following form (Grandell, Segerdahl 1971):

$$\psi(u) = \frac{\theta\left(1-\frac{R}{\alpha}\right) exp\left(\frac{-\beta R u}{\alpha}\right)}{1+(1+\theta)R-(1+\theta)\left(1-\frac{R}{\alpha}\right)} + \frac{\alpha\theta\, sin(\alpha\pi)}{\pi}\cdot I, \quad \text{where:}$$

$$I = \int_0^\infty \frac{x^\alpha exp\{-(x+1)\beta u\}}{[x^\alpha\{1+\alpha(1+\theta)(x+1)\}-cos(\alpha\pi)]^2+sin^2(\alpha\pi)}dx. \tag{16}$$

The above integral is calculated numerically. Moreover, from the properties of the gamma distribution, we are able to adjust the formula for any mean, and thus liberalize the assumption of equality of the two parameters (Burnecki, Miśta, Weron 2005a):

$$\psi_X(u) = \psi_{X/\mu}(u/\mu), \qquad \psi_{G(\alpha,\beta)}(u) = \psi_{G(\alpha,\alpha)}(\beta u/\alpha). \tag{17}$$

Modifying both parameters allows obtaining a whole range of distributions with different shapes and expected values. Ruin probabilities for sample values are presented below:

Table 3. Ruin probability depending on the initial surplus and safety loading coefficient for a gamma distribution with parameters $\alpha = 0.5$, $\beta = 0.0005$

	$u = 0$	$u = 10^3$	$u = 5 \cdot 10^3$	$u = 10^4$	$u = 2{,}5 \cdot 10^4$	$u = 5 \cdot 10^4$	$u = 10^5$
$\theta = 0.05$	0.9524	0.9191	0.8088	0.6907	0.4301	0.1953	0.0403
$\theta = 0.10$	0.9091	0.8495	0.6662	0.4935	0.2008	0.0448	0.0022
$\theta = 0.15$	0.8695	0.7890	0.5575	0.3632	0.1006	0.0118	0.00016
$\theta = 0.20$	0.8333	0.7361	0.4730	0.2743	0.0536	0.0035	0.00001
$\theta = 0.25$	0.8000	0.6894	0.4062	0.2119	0.0301	0.0012	0.000002
$\theta = 0.30$	0.7692	0.6480	0.3527	0.1669	0.0177	0.0004	$<10^{-7}$

Source: Own study.

The values of the parameters in Table 3 were chosen so that the expected value of the loss amount is 1000 – in this way they can be compared in some way with the exponential distribution shown in Table 1. An obvious fact from the definition of the expected value in the gamma distribution is the observation that simultaneously increasing the parameter α and decreasing the parameter β will generate the highest possible loss achievable in this class of distributions. Table 4 presents a numerical example showing the limit case for the shape parameter $\alpha = 1$, and scale parameter $\beta = 10^{-7}$ – resulting in an expected value of loss amount equal to 1 000 000.

Table 4. Ruin probability depending on the initial surplus and safety loading coefficient for a gamma distribution with parameters $\alpha = 1, \beta = 10^{-7}$

	$u = 0$	$u = 10^5$	$u = 10^6$	$u = 5 \cdot 10^6$	$u = 10^7$	$u = 2 \cdot 10^7$	$u = 5 \cdot 10^7$
$\theta = 0.05$	0.9524	0.9479	0.9081	0.7506	0.5916	0.3674	0.0881
$\theta = 0.10$	0.9091	0.9009	0.8301	0.5770	0.3663	0.1476	0.00965
$\theta = 0.15$	0.8695	0.8583	0.7632	0.4530	0.2360	0.0640	0.00127
$\theta = 0.20$	0.8333	0.8196	0.7054	0.3622	0.1574	0.0297	0.0002
$\theta = 0.25$	0.8000	0.7842	0.6550	0.2943	0.1083	0.0147	0.00003
$\theta = 0.30$	0.7692	0.7517	0.6107	0.2426	0.0765	0.0076	0.000007

Source: Own study.

2. A comparison of different approximation methods

An overview of selected approximation methods is presented in Table 5. Comparing their quality over an infinite time horizon can be troublesome due to the lack of a reliable benchmark. Explicit formulae for exact probabilities are known only in specific cases; in all other cases, reliable evaluation of the results of a given approximation is difficult because we do not know the actual ruin probabilities. One way to deal with this issue is to assume a very long time horizon T, thus imitating infinity and relying on the results of Monte Carlo simulations, however this will still be subject to errors, as well as consuming a lot of computing power. Another better alternative is to use the Pollaczek-Khinchin theorem, which uses the notion of aggregate loss, allowing the ruin probability to be determined numerically (Grandell 2000). The following section will present the results of the various approximation methods for different distributions of the loss amount (divided into light-tailed and heavy-tailed), along with a discussion and comparison against the background of the Pollaczek-Khinchin method. A constant safety loading coefficient of 0.25 was used in all discussed cases. An overview of used distributions can be found in Table 6.

Table 5. An overview of chosen methods of approximating ruin probability

Type	Approximation	Formula	Description
Direct approximation of the ruin probability function	Cramer--Lundberg (abbreviation: C-L)	$\psi_{\mathrm{CL}}(u) = \frac{\theta\mu}{M'_x(R) - (1+\theta)\mu} e^{-Ru}$	General use, especially good results for huge initial surplus values
	Zero	$\psi_0(u) = \frac{1}{(1+\theta)} e^{-Ru}$	Naive method, equal results for zero initial surplus
	Exponential	$\psi_{\mathrm{E}}(u) = \exp\left(-1 - \frac{2\theta\mu u - \mu^{(2)}}{\sqrt{(\mu^{(2)})^2 + \left(\frac{4}{3}\right)\theta\mu\mu^{(3)}}}\right)$	Existence of first three ordinary moments: $\mu, \mu^{(2)}, \mu^{(3)}$ of the loss amount distribution, no reliance on adjustment coefficient – applicability to heavy-tailed distributions.
Approximation of the conditional distribution of the total maximum loss	Beekman-Bowers (abbreviation: B-B)	$\psi_{\mathrm{BB}}(u) = \frac{1}{(1+\theta)}\{1 - F_{Gamma(\alpha,\beta)}(u)\}$ $\alpha = \frac{\left[1+\left(\frac{4\mu\mu^{(3)}}{3\left(\mu^{(2)}\right)^2}-1\right)\theta\right]}{(1+\theta)}, \quad \beta = \frac{2\theta\mu}{\left[\mu^{(2)}+\left(\frac{4\mu\mu^{(3)}}{3\mu^{(2)}}-\mu^{(2)}\right)\theta\right]}$	Existence of first three ordinary moments: $\mu, \mu^{(2)}, \mu^{(3)}$ of the loss amount distribution, no reliance on adjustment coefficient. Large error if the fourth moment does not exist (although it is not formally required for the method)
	Renyi	$\psi_{\mathrm{R}}(u) = \frac{1}{(1+\theta)} \exp\left\{-\frac{2\theta\mu u}{\mu^{(2)}(1+\theta)}\right\}$	Simplified version of the B-B approximation. Existence of first three ordinary moments: $\mu, \mu^{(2)}, \mu^{(3)}$ of the loss amount distribution

Table 5 – continued

Type	Approxi-mation	Formula	Description
Approximation of surplus process growth	De Vylder	$\psi_{DV}(u) = \frac{1}{1+\hat{\theta}} e^{\left(\frac{-\hat{\theta}\hat{\beta}u}{1+\hat{\theta}}\right)}$, gdzie $\hat{\beta} = \frac{3\mu^{(2)}}{\mu^{(3)}}, \hat{\lambda} = \frac{9\lambda{\mu^{(2)}}^3}{2{\mu^{(3)}}^2}, \hat{\theta} = \frac{2\mu\mu^{(3)}\theta}{3{\mu^{(2)}}^2}$	Existence of first three ordinary moments: $\mu, \mu^{(2)}, \mu^{(3)}$ of the loss amount distribution, no reliance on adjustment coefficient – applicability to heavy-tailed distributions.
	4-gamma De Vylder	$\psi(u) = \frac{\hat{\theta}\left(1-\frac{R}{\hat{\alpha}}\right)\exp\left(\frac{-\hat{\beta}Ru}{\hat{\alpha}}\right)}{1+\left(1+\hat{\theta}\right)R-\left(1+\hat{\theta}\right)\left(1-\frac{R}{\hat{\alpha}}\right)} +$ $+\frac{\hat{\alpha}\hat{\theta}\sin(\hat{\alpha}\pi)}{\pi}\cdot I$, where $\widehat{\lambda} = \frac{\lambda(\mu^{(3)})^2(\mu^{(2)})^3}{(\mu^{(2)}\mu^{(4)}-2(\mu^{(3)})^2)(2\mu^{(2)}\mu^{(4)}-3(\mu^{(3)})^2)}$ $\widehat{\theta} = \frac{\theta\mu[2(\mu^{(3)})^2-\mu^{(2)}\mu^{(4)}]}{(\mu^{(2)}\mu^{(3)})^2}$ $\widehat{\mu} = \frac{3(\mu^{(3)})^2-2\mu^{(2)}\mu^{(4)}}{\mu^{(2)}\mu^{(3)}}$ $\widehat{\mu^{(2)}} = \frac{(\mu^{(2)}\mu^{(4)}-2\left(\mu^{(3)}\right)^2)(2\mu^{(2)}\mu^{(4)}-3\left(\mu^{(3)}\right)^2)}{(\mu^{(2)}\mu^{(3)})^2}$ $I = \int_0^\infty \frac{x^{\hat{\alpha}}\exp\{-(x+1)\hat{\beta}u\}}{[x^{\hat{\alpha}}\{1+\hat{\alpha}\left(1+\hat{\theta}\right)(x+1)\}-\cos(\hat{\alpha}\pi)]^2+sin^2(\hat{\alpha}\pi)}dx,$ and $\hat{\alpha} = \frac{\hat{\mu}^2}{\widehat{\mu^{(2)}}-\hat{\mu}^2}$ i $\hat{\beta} = \frac{\hat{\mu}}{\widehat{\mu^{(2)}}-\hat{\mu}^2}$	Existence of first four ordinary moments: $\mu, \mu^{(2)}, \mu^{(3)}, \mu^{(4)}$ of the loss amount distribution, no reliance on adjustment coefficient – applicability to heavy-tailed distribution.

Source: Own study based on Grandell 2000; Tura 2015; Burnecki, Weron, Miśta 2005a.

Table 6. Overview of chosen probability distributions

Distribution type	Distribution	Parameters	Probability density function
Light-tailed	Exponential	$\beta > 0$	$f_X(x) = \beta exp\,(-\beta x)$
	Mixture of exponential distributions	$\beta_i > 0, \sum_{i=1}^{n} \alpha_i = 1$	$f_X(x) = \sum_{i=1}^{n} \{\alpha_i \beta_i exp\,(-\beta_i x)\}$
	Gamma	$\alpha > 0, \beta > 0$	$f_X(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} exp\,(-\beta x)$
Heavy-tailed	Pareto	$\alpha > 0, \beta > 0$	$f_X(x) = \frac{\alpha}{\beta + x}(\frac{\beta}{\beta + x})^\alpha$
	Lognormal	$\mu \in \mathbb{R}, \sigma > 0$	$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma x} exp\,\{-\frac{(\ln x - \mu)^2}{2\sigma^2}\}$
	Weibull	$\beta > 0, 0 < \tau < 1$	$f_X(x) = \beta\tau x^{\tau-1} exp\,(-\beta x^\tau)$
	Burr	$\alpha > 0, \sigma > 0, \tau > 1$	$f_X(x) = \frac{\alpha\tau\lambda^\alpha x^{\tau-1}}{(\lambda + x^\tau)^{\alpha+1}}$

Source: Burnecki, Teurle, Wilkowska 2019.

2.1. Light-tailed distributions – mixture of exponential distributions, gamma distribution

Table 7. Ruin probability depending on the initial surplus and approximation method for a mixture of exponential distributions with parameters $\alpha = 5 \cdot 10^{-10}$, $\beta = 7.5 \cdot 10^{-9}$, respective weights: 0.1, 0.9

	$u = 0$	$u = 10^9$	$u = 5 \cdot 10^9$	$u = 10^{10}$	$u = 2 \cdot 10^{10}$	$u = 5 \cdot 10^{10}$
Exact result	0.8000	0.6313	0.3605	0.1791	0.0442	$6.64 \cdot 10^{-4}$
Cramer – Lundberg	0.7257	0.6309	0.3604	0.1791	0.0442	$6.64 \cdot 10^{-4}$
Zero	0.8000	0.6955	0.3974	0.1974	0.0487	$7.32 \cdot 10^{-4}$
Exponential	0.7849	0.6784	0.3788	0.1828	0.0426	$5.38 \cdot 10^{-4}$
Beekman – Bowers	0.8000	0.6560	0.3540	0.1738	0.0442	$8.17 \cdot 10^{-4}$
Renyi	0.8000	0.6859	0.3707	0.1718	0.0369	$3.65 \cdot 10^{-4}$
De Vylder (4-gamma)	0.7434	0.6364	0.3604	0.1789	0.0442	$6.65 \cdot 10^{-4}$

Source: Own study.

Table 8. Relative error for the data in Table 7 (%)

	$u = 0$	$u = 10^9$	$u = 5 \cdot 10^9$	$u = 10^{10}$	$u = 2 \cdot 10^{10}$	$u = 5 \cdot 10^{10}$
Exact result	0	0	0	0	0	0
Cramer – Lundberg	-9.29	-0.06	-0.02	<-0.01	<-0.01	<-0.01
Zero	0	10.18	10.24	10.24	10.24	10.24
Exponential	-1.89	7.47	5.07	2.07	-3.68	-19.06
Beekman – Bowers	0	3.92	-1.81	-2.94	-1.68	22.96
Renyi	0	8.66	2.83	-4.08	-16.55	-45.04
De Vylder (4-gamma)	-7.07	0.82	-0.02	0.08	<-0.01	-0.02

Source: Own study.

For the given parameters, the expected value of the distribution is:

$$EY = \frac{0.10}{5 \cdot 10^{-10}} + \frac{0.90}{7.5 \cdot 10^{-9}} = 3.2 \cdot 10^8.$$

The range of initial surplus thus corresponds to a range of approximately: 3–150 times the amount of a single loss with arbitrary points representing the capital of the insurance company, where a noticeable decrease in consecutive ruin probabilities is successively recorded. The analysis of the acceptable relative error for different levels of initial surplus remains subjective – its sensitivity is variable and depends on the results of the absolute ruin probability – when it is small (less than 1%), then it is very easy to have relatively large deviations. Analysing Table 7 and 8, several conclusions can be drawn: apart from zero initial surplus, the Cramer-Lundberg and 4-gamma De Vylder approximations gave excellent results, with a relative error not exceeding 1% in either direction. In case of the C-L approximation, it is known to converge monotonically from the left to the exact result, so it will always minimally underestimate the actual results which may allow the use of a scaling factor (especially for low values of initial surplus) to get even closer to the real probability. For large values of initial surplus, the method is definitely unbeatable for a mixture of exponential distributions. Of the methods based on the conditional distribution of the total maximum loss, the Beekman-Bowers approximation also gives very satisfactory results, in each case beating its simplified version – the Renyi approximation. The Zero method, unlike the Lundberg approximation, gives an accurate result with zero initial surplus, with a consistent increase in error that seems to converge to a specific value. The possibility to estimate the maximum possible error of a given technique is also useful information, so this method – despite its relatively large deviations – can provide an interesting reference point with other parameter values.

Table 9. Ruin probability depending on the initial surplus and approximation method with parameters α = 0.25, *β* = 0.000003

	$u = 0$	$u = 10^6$	$u = 2 \cdot 10^6$	$u = 3 \cdot 10^6$	$u = 4 \cdot 10^6$	$u = 5 \cdot 10^6$
Exact result	0.8	0.3038	0.1212	0.0484	0.0193	$7.71 \cdot 10^{-3}$
Cramer – Lundberg	0.7602	0.3035	0.1212	0.0484	0.0193	$7.71 \cdot 10^{-3}$
Zero	0.8	0.3194	0.1275	0.0509	0.0203	$8.12 \cdot 10^{-3}$
Exponential	0.8110	0.3141	0.1216	0.0471	0.0182	$7.06 \cdot 10^{-3}$
Beekman – Bowers	0.8	0.3018	0.1200	0.0483	0.0195	$7.91 \cdot 10^{-3}$
Renyi	0.8	0.3063	0.1173	0.0449	0.0172	$6.58 \cdot 10^{-3}$
De Vylder (4-gamma)	0.8	0.3038	0.1212	0.0484	0.0193	$7.71 \cdot 10^{-3}$

Source: Own study.

Table 10. Relative error for the data in Table 9 (%)

	$u = 0$	$u = 10^6$	$u = 2 \cdot 10^6$	$u = 3 \cdot 10^6$	$u = 4 \cdot 10^6$	$u = 5 \cdot 10^6$
Exact result	0	0	0	0	0	0
Cramer – Lundberg	-4.98	-0.08	<-0.01	<-0.01	<-0.01	<-0.01
Zero	0	5.15	5.23	5.23	5.24	5.24
Exponential	1.38	3.40	0.37	-2.65	-5.57	-8.41
Beekman – Bowers	0	-0.64	-0.95	-0.24	1.00	2.59
Renyi	0	0.84	-3.22	-7.18	-10.98	-14.76
De Vylder (4-gamma)	0	0	0	0	0	0

Source: Own study.

For the gamma distribution with parameters used in Tables 9 and 10, the expected value of the loss amount is

$$EY = \frac{0.25}{3 \cdot 10^{-6}} = 83\,333.(3),$$

which, with the initial surplus values used, gives a range: 12–60 times the value of a single loss. The unbeatable method in this case is De Vylder 4-gamma approximation, because with the loss amount following a gamma distribution we get exactly the same values – the new process approximated by the gamma distribution is in fact a gamma distribution from the very beginning, so the obvious (and expected) conclusion is to obtain equality. The other approximations behave very similarly to the mixture of exponential distributions – great approximations are again obtained by the C-L method, the error obtained with the Zero approximation has almost decreased by half, the B-B approximation also performs very well, still being better than the Renyi approximation for any initial surplus value. When the actual ruin probability is greater than 2%, basically all methods give acceptable results, with rare cases of a relative error of more than 5% – this being the case only for the Renyi approximation (where in case of a loss amount given by a gamma distribution it is recommended to use the B-B method) and the Zero method, which however is subject to a consistent overestimation error, possible to be corrected by an appropriate scaling factor due to the convergence of the error with increasing initial surplus.

2.2. Heavy-tailed distributions – Pareto distribution, lognormal distribution, Weibull distribution, Burr distribution

In case of heavy-tailed distributions, the scheme for presenting information remains identical, but the reference point is changed to the Pollaczek-Khinchin method, due to the lack of analytical formulae and the inability to determine the exact ruin probability. To generate results using this technique, 50 blocks of 50 000 simulations (2 500 000 simulations in total) were used in each of the discussed loss amount distributions.

Table 11. Ruin probability depending on the initial surplus and approximation method for the Pareto distribution with parameters $\alpha = 4.2$, $\beta = 10^9$

	$u = 0$	$u = 10^9$	$u = 2 \cdot 10^9$	$u = 5 \cdot 10^9$	$u = 10^{10}$	$u = 2 \cdot 10^{10}$
Pollaczek – Khinchin	0.8	0.4805	0.3115	0.095	0.0158	$8.34 \cdot 10^{-4}$
Exponential	0.7575	0.5092	0.3422	0.1039	0.0142	$2.68 \cdot 10^{-4}$
Beekman – Bowers	0.8	0.4694	0.3104	0.1000	0.0168	$5.36 \cdot 10^{-4}$
Renyi	0.8	0.5152	0.3318	0.0886	0.0010	$1.21 \cdot 10^{-4}$
De Vylder (4-gamma)	0.721	0.4680	0.3192	0.1036	0.0160	$3.83 \cdot 10^{-4}$

Source: Own study.

Table 12. Relative error for the data in Table 11 (%)

	$u = 0$	$u = 10^9$	$u = 2 \cdot 10^9$	$u = 5 \cdot 10^9$	$u = 10^{10}$	$u = 2 \cdot 10^{10}$
Pollaczek – Khinchin	0	0	0	0	0	0
Exponential	-5.31	5.98	9.88	9.38	-9.75	-67.80
Beekman – Bowers	<0.01	-2.30	-0.33	5.20	6.56	-35.68
Renyi	<0.01	7.24	6.53	-6.71	-37.83	-85.53
De Vylder (4-gamma)	-9.87	-2.60	2.48	9.06	1.35	-54.09

Source: Own study.

Tables 11 and 12 analyse an example of a Pareto distribution with parameter values: $\alpha = 4.2$, $\beta = 10^9$ the expected value of the loss amount was

$$EY = \frac{4.2 \cdot 10^9}{4.2 - 1} = 1.31 \cdot 10^9.$$

The C-L and Zero methods are not usable due to the fact that the MGF does not exist; the 4-gamma De Vylder, on the other hand, requires a finite fourth moment, hence the usage of values above 4 for the shape parameter. The best-looking approximation is the B-B method, which for single-digit ruin probability values only slightly exceeds the 5% error tolerance. 4-gamma De Vylder method also performs satisfactorily. In a comparative analysis with light-tailed distributions, the relative error at low ruin probabilities increases very significantly. Moreover, due to the unavailability of the C-L method, we do not have the tools to eliminate (at least asymptotically) the estimation error. Therefore, considering the need to study ruin at a very high probability quantile (e.g., for Solvency II at 99.5%), the only reliable indicator for the loss amount given by the Pareto distribution is the Pollaczek-Khinchin method, followed (most likely) by the B-B approximation.

Table 13. Ruin probability depending on the initial surplus and approximation method for the lognormal distribution with parameters $\mu = 19$, $\sigma = 1.03$

	$u = 0$	$u = 10^9$	$u = 2 \cdot 10^9$	$u = 5 \cdot 10^9$	$u = 10^{10}$	$u = 2 \cdot 10^{10}$
Pollaczek – Khinchin	0.7995	0.4631	0.2979	0.0907	0.0154	$8.59 \cdot 10^{-4}$
Exponential	0.7511	0.4998	0.3327	0.0980	0.0128	$2.18 \cdot 10^{-4}$
Beekman – Bowers	0.8	0.4576	0.2994	0.0944	0.0155	$4.72 \cdot 10^{-4}$
Renyi	0.8	0.5068	0.3211	0.0817	0.0083	$8.69 \cdot 10^{-4}$
De Vylder (4-gamma)	0.7119	0.4568	0.3090	0.0980	0.0146	$3.24 \cdot 10^{-4}$

Source: Own study.

Table 14. Relative error for the data in Table 13 (%)

	$u = 0$	$u = 10^9$	$u = 2 \cdot 10^9$	$u = 5 \cdot 10^9$	$u = 10^{10}$	$u = 2 \cdot 10^{10}$
Pollaczek – Khinchin	0	0	0	0	0	0
Exponential	-6.06	8.32	11.67	8.11	-16.74	-74.57
Beekman – Bowers	0.06	-1.19	0.52	4.03	0.68	-44.94
Renyi	0.06	9.44	7.80	-9.97	-45.79	-89.88
De Vylder (4-gamma)	-10.95	-1.36	3.72	8.07	-5.11	-62.28

Source: Own study.

In the analyzed lognormal distribution example, the parameters are: $\mu = 19$, $\sigma = 1.03$, which gives an expected value approximately equal to: $EY \approx 3 \cdot 10^8$. A comparison of the results in Table 13 and 14 generally leads to similar conclusions as for the Pareto distribution. The best method still seems to be the B-B approximation, which for ruin probability in the interval of 2–80% gives very stable and satisfactory results, with a relative error of no more than 5%. The four-moment gamma De Vylder method still performs satisfactorily, although just slightly less so. Once again, it can be seen that it underestimates the probabilities at the beginning of the range of initial surplus volatility. This is not a critical flaw, however, given that the strict focus is on low values of ruin probability – and setting the level of premiums that allows bankruptcy to occur at a very low but realistic and acceptable level. Exponential approximation and Renyi seem to yield similar results, with several times the error of the best methods in the example (B-B and De Vylder 4-gamma). As with the Pareto distribution, at the very tail of the distribution, for ruin probabilities of the order of the fourth decimal place, all methods record a significant relative error, consistently underestimating the real risk of bankruptcy. From a prudential point of view, therefore, they should not be an alternative to the indications obtained by the Pollaczek-Khinchin method, and possible only provide additional information.

Table 15. Ruin probability depending on the initial surplus and approximation method for the Weibull distribution with parameters $\beta = 1$, $\tau = 0.5$

	$u = 0$	$u = 5$	$u = 10$	$u = 25$	$u = 50$	$u = 100$
Pollaczek – Khinchin	0.8001	0.4854	0.2945	0.0658	0.0054	$3.68 \cdot 10^{-5}$
Exponential	0.8323	0.4997	0.3000	0.0649	0.0050	$3.07 \cdot 10^{-5}$
Beekman – Bowers	0.8	0.4852	0.2943	0.0657	0.0054	$3.63 \cdot 10^{-5}$
Renyi	0.8	0.4852	0.2943	0.0657	0.0054	$3.63 \cdot 10^{-5}$
De Vylder (4-gamma)	0.8	0.4852	0.2943	0.0657	0.0054	$3.63 \cdot 10^{-5}$

Source: Own study.

Table 16. Relative error for the data in Table 15 (%)

	u = 0	*u* = 5	*u* = 10	*u* = 25	*u* = 50	*u* = 100
Pollaczek – Khinchin	0	0	0	0	0	0
Exponential	4.03	2.93	1.85	-1.33	-5.59	-16.46
Beekman – Bowers	-0.01	-0.04	-0.06	-0.15	0.59	-1.32
Renyi	-0.01	-0.04	-0.06	-0.15	0.59	-1.32
De Vylder (4-gamma)	-0.01	-0.04	-0.06	-0.15	0.59	-1.32

Source: Own study.

The Weibull distribution has a heavy tail when the shape parameter is in the interval (0, 1). In the considered example, the parameters of the distributions are: $\beta = 1$ and $\tau = 0.5$, which gives an expected value equal to 2. The summary of results in Tables 15 and 16 strongly suggests that in principle all the discussed approximation methods are acceptable for use. The B-B, Renyi and De Vylder methods give almost exact results, the relative error for them is practically minimal, moreover, no significant differences can be seen between any of the aforementioned techniques. Exponential approximation also estimates the probability very well against the results obtained with the Pollaczek-Khinchin algorithm. The relative error increases above the arbitrary threshold of 5% only for ruin probabilities of the order of the third decimal place. Moreover, all of the presented methods performed much better than the light-tailed distributions, which raises the question of a possible reason for this phenomenon. The answer may lie in the similarity between the Weibull distribution and the exponential distribution with the parameters used in the tables. The Weibull distribution, depending on the assumed shape parameter λ, allows a wide range of distributions: similar to normal (for large λ), for $\lambda = 1$ it reducees to an exponential distribution, and for $\lambda < 1$, we get a thicker tail than in the exponential case, but still the overall shape and behavior of the distribution is strongly related to that of the exponential distribution. All approximation methods seek, to some extent, to make the loss amount distribution similar to the exponential distribution or to fit an appropriate number of moments, so intuitively increasing similarity suggests better approximation results. It is worth to note that the noticeable differences occur even though formally the Weibull distribution with $k = 0.5$ is a heavy-tailed distribution, which should inherently be more difficult to estimate than any light-tailed distribution.

The last discussed distribution will be the Burr distribution with parameters: $\alpha > 0$, $\sigma > 0$, $\tau > 0$. The family of Burr distributions is a flexible group of distributions with heavy tails. To use methods based on third and fourth moments, it is necessary that the product $\alpha \cdot \sigma$ is greater than 4. In the analyzed example, the values are: $\alpha = 2.5$; $\sigma = 1.65$; $\tau = 8$.

Table 17. Ruin probability depending on the initial surplus and approximation method for the Burr distribution with parameters α = 2.5, σ =1.65, τ = 8

	$u = 0$	$u = 5$	$u = 10$	$u = 25$	$u = 50$	$u = 100$
Pollaczek – Khinchin	0.8000	0.4785	0.2933	0.0725	$8.03 \cdot 10^{-3}$	$2.08 \cdot 10^{-4}$
Exponential	0.8014	0.4984	0.3100	0.0746	$6.94 \cdot 10^{-3}$	$6.02 \cdot 10^{-5}$
Beekman – Bowers	0.8000	0.4740	0.2951	0.0748	$7.99 \cdot 10^{-3}$	$9.59 \cdot 10^{-5}$
Renyi	0.8000	0.4911	0.3015	0.0670	$6.08 \cdot 10^{-3}$	$4.62 \cdot 10^{-5}$
De Vylder (4-gamma)	0.7616	0.4757	0.3001	0.0756	$7.58 \cdot 10^{-3}$	$7.64 \cdot 10^{-5}$

Source: Own study.

Table 18. Relative error for the data in Table 17 (%)

	$u = 0$	$u = 5$	$u = 10$	$u = 25$	$u = 50$	$u = 100$
Pollaczek – Khinchin	0	0	0	0	0	0
Exponential	0.17	4.17	5.71	2.95	-13.51	-71.02
Beekman – Bowers	<-0.01	-0.93	0.64	3.27	-0.51	-53.80
Renyi	<-0.01	2.64	2.81	-3.73	-24.24	-77.73
De Vylder (4-gamma)	-4.81	-0.59	2.34	4.27	-5.53	-63.19

Source: Own study.

The results shown in Tables 17 and 18 seem consistent with other results obtained so far. They are not as spectacular as in the case of the Weibull distribution, but are nevertheless still within acceptable tolerance thresholds, and show better properties than in the case of, e.g. Pareto or lognormal distributions. Once again, the B-B approximation yields the closest results to reality, confirming why it is thought to be the "best of the simplest" ruin probability approximations. In addition to its very satisfactory results, its application is not limited to light-tailed distributions, so moving onto the collective drawing of conclusions based on all analyzed distributions, it seems to be the method recommended for use in the first place (if only possible due to a finite third moment). Slightly lower in the recommendation are the De Vylder and Renyi approximations, with a numerical advantage for the former, although it should be remembered that, relative to Renyi, it requires the existence of two more moments (third and fourth) and therefore its use is subject to additional conditions on the distribution of the loss amount.

For certain applications, in particular a light-tailed distribution and high initial surplus (compared to the expected value of a single loss), the C-L approximation gives best results. An additional advantage of this method is the deterministic knowledge of the *direction* of the error. While in practice a high initial surplus is an assumption that is very often satisfied, the lack of applicability for heavy-tailed distributions is a rather serious shortcoming of this approximation. Nevertheless, it should be noted that (with a few exceptions) most of the methods gave decent results, at least for fairly reasonable limits of variability of the actual ruin probability, not exceeding 2–5%. From a reporting point of view, however, this is too low a confidence level, and further increasing quantiles inevitably reduces the ability to control the relative error with almost all approximations.

Finally, it is also worth referring to the results of similar analyses of the quality of approximation methods for ruin probability (Grandel, Segerdahl 1971) (Grandell 2000) (Burnecki, Miśta, Weron 2005a). The analyses indicate a relatively high quality of fit of De Vylder and Beekman-Bowers approximations, which is consistent with the conclusions drawn in this article. More attention has also been paid to the De Vylder approximation in (Burnecki, Miśta, Weron 2005b), presenting the approximation method based on the first four moments as an improvement over the traditional version. The version based on four moments was also used in this article. An extensive comparison of approximation methods was also made in (Burnecki, Miśta, Weron 2005c), where the use of the De Vylder approximation (including the version based on four moments), in addition to the Beekman-Bowers, among others, led to the relatively smallest errors. In this context, the results obtained in this article remain consistent with those obtained previously by other researchers.

2.3. Assessment of the feasibility of using different approximation methods

The article presents various methods of approximating the ruin probability and analyzes the quality of these approximations. The possibility of their use by insurance companies in their operations remains a separate issue. One area of application is pricing, where the insurance premium is defined in such a way that the ruin probability does not exceed a set level. Having at its disposal assumptions about the process of occurrence of claims (both the distribution of the number of claims over a period of time and the distribution of single loss amount), an insurance company can use the proposed approximation methods to set the premium associated with a fixed ruin probability as precisely as possible. The quality of the fit plays a key role here, as underestimating the ruin probability can lead to insolvency of the company, while overestimating reduces competitive advantage. Of course, due to the multiplicity of insurance products, the volatility of economic conditions, customer behavior, regulatory restrictions and a number of other factors, the applicability of ruin theory may be limited. Nevertheless, the practical usefulness of this type of analysis may arise in comparative analysis, e.g. at certain stages of the development of an insurance offering, when comparing different variants of a potential offering. The second area of application of approximation methods is the estimation of the ruin probability for an existing portfolio of insurance contracts. A high value of this probability may mean that the company is exposed to excessive insurance risk and may consider taking measures to reduce this risk (such as reinsurance). The application of ruin theory and the De Vylder approximation to a two-dimensional model (including reinsurance) is further discussed by K. Burnecki, M. Teuerle and A. Wilkowska (Burnecki, Teuerle, Wilkowska 2019). Ruin theory can also be used to optimize dividend payment policy and more broadly in the context of the Solvency II regime (Loisel, Gerber 2012). In this context, the model can be redefined so that ruin occurs when the SCR coverage ratio (ratio of own funds to the solvency capital requirement) falls below 100%. Although this does not mean bankruptcy for the insurance company, it is an unfavorable situation, as the company does not have sufficient funds to ensure that the probability of insolvency over a one-year horizon remains below 0.5%.

Regardless of the purpose of an insurance company, it may not be possible to accurately determine the ruin probability and therefore it is necessary to use approximations. For light-tailed distributions, a good quality fit for low values of ruin probability is obtained when using the Cramer-Lundberg approximation or the De Vylder approximation. For heavy-tailed distributions, the Beekman-Bowers approximation turns out to be better. The company's decision on the appropriate method will therefore depend on whether it expects a light-tailed or a heavy-tailed distribution for a particular insurance contract. The parameters necessary to apply the approximation, in particular the moments of the distribution of the loss amount, can be estimated from historical data. Determining the limiting ruin probability also remains an important issue. A level of 0.5% may be a fairly natural choice, as it corresponds to the Solvency II Directive's method of calculating the Solvency Capital Requirement (SCR). However, it should be noted that this requirement corresponds to the value at risk for the entire company over a one-year horizon, which is not consistent wit estimating the ruin probability for a specific group of contracts over an infinite time horizon. However, one can consider using an approximation to estimate the probability of ruin over a finite horizon, such as one year. It is also possible, as noted earlier, to use a model in which ruin means that the SCR coverage ratio falls below 100%, which will allow the insurance company to make some kind of a projection of the Solvency II regime's requirements into the future and answer the question of what the risk of not meeting the SCR coverage ratio requirement is in the long term.

Conclusion

This paper presents the theoretical basis of ruin theory, the classical model and exact results for selected probability distributions of single loss amount (where ruin probability can be determined analytically), and analyzes the quality of available approximation methods for selected distributions (both light- and heavy-tailed) depending on the values of the parameters describing the surplus process.

The quality of approximation is affected by a number of factors, one of the key ones under the control of the insurance company being the level of initial surplus. Depending on this level, the quality of approximation methods, measured as the difference between the estimated and actual ruin probabilities, can vary (as the amount of initial surplus increases, absolute approximation errors decrease, which is associated with a decrease in the ruin probability, however relative errors tend to increase). In the case of light-tailed distributions, the lowest relative estimation errors were observed for the Cramer-Lundberg approximation (which, compared to the other considered methods, can be considered relatively simplified) and the De Vylder approximation (which has the most complex form among the analyzed approximation methods). For heavy-tailed distributions, the Beekman-Bowers approximation has the advantage, although relatively low errors were also observed for the De Vylder approximation. Overall, these methods are fairly good approximations of the actual ruin probability and can be successfully used in insurance company calculations.

Although ruin theory is primarily a theoretical concept that facilitates the understanding of the risks faced by an insurance company, it can be applied to the quantification of risk for risk management purposes, and can also be used as a supporting tool in the calculation of insurance premiums (where the goal is to set the premium at such a level that the ruin probability does not exceed a set critical value). It is also impossible to ignore the peculiar interactions between ruin theory and the requirements of the Solvency II regime. When defining ruin as a situation in which the SCR coverage ratio falls below 100%, the insurance company is given an interesting tool for analyzing solvency over a long-time horizon (going beyond the one-year framework defined in the calculation of the SCR). In all these areas, an exact calculation of the ruin probability may not be possible, and approximations may be necessary. The theory can also be applied to non-insurance areas, such as option pricing (Gerber, Shiu 1999). Understanding ruin approximation methods, their quality and limitations allows ruin theory to be used more effectively to analyze actual claims processes in an insurance company and in other areas where the theory is applicable.

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